JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 19, No. 2, March–April 1996

Frequency Domain Performance Bounds for Uncertain Positive Real Systems

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An important part of feedback control involves analyzing uncertain systems for robust stability and performance. In certain applications, such as collocated control of flexible structures, the plant is positive real. Hence, closed-loop stability is unconditionally guaranteed as long as the controller is also positive real. Most of the performance robustness analysis results in the literature, however, will not always yield finite performance bounds for the case of closed-loop systems consisting of uncertain positive real plants controlled by strictly positive real compensators. These results are obviously conservative since this class of systems is unconditionally stable. Majorant analysis is used to develop tests that yield finite frequency-dependent performance bounds for the described case. The results are specialized to the case of static, decentralized collocated rate feedback and dynamic, collocated rate feedback. These results are compared to previous majorant performance bounds and performance bounds resulting from real structured singular value analysis.

Nomenclature

$\det(Z)$	= determinant of square matrix Z
$\mathrm{diag}(Z_1, \ldots, Z_n)$	= diagonal matrix with listed diagonal elements
$\mathrm{He}\, Z$, $\mathrm{Sh}\, Z$	= $\frac{1}{2}(Z + Z^*)$, $\frac{1}{2}(Z - Z^*)$
$\mathcal{L}[z(t)]$	= Laplace transform of $z(t)$
$\mathbb{R}$, $\mathbb{C}$, I_p	= set of real numbers, set of complex numbers, $p \times p$ identity matrix
$\|x\|_2$	= Euclidean norm of vector x, equal to $\sqrt{(x^*x)}$
$Y \leq\leq Z (Y \ll Z)$	= $Y_{(i,j)} \leq Z_{(i,j)} (Y_{(i,j)} < Z_{(i,j)})$ for each i and j, where Y and Z are real matrices with identical dimensions
Z_d	= diagonal part of a square matrix Z, equal to $\mathrm{diag}(Z_{(1,1)}, \ldots, Z_{(n,n)})$
$\|Z\|_F$	= Frobenius norm of matrix Z, equal to $(\mathrm{tr}\, ZZ^*)^{1/2}$
Z_{od}	= off-diagonal part of a square matrix Z, equal to $Z - Z_d$
$Z_{(i,j)}$	= (i, j) element of matrix Z
$\|Z\|_s$	= spectral norm of matrix Z, equal to $\sigma_{\max}(Z)$
Z^*, Z^{-*}	= complex conjugate transpose of matrix Z, $(Z^*)^{-1}$
$\|\alpha\|$	= absolute value of complex scalar α
$\lambda_{\min}(Z)$, $\lambda_{\max}(Z)$	= minimum, maximum eigenvalues of the Hermitian matrix Z
$\rho(Z)$	= spectral radius of a square matrix Z
$\sigma_{\min}(Z)$, $\sigma_{\max}(Z)$	= minimum, maximum singular values of matrix Z

I. Introduction

IN certain applications, such as the control of flexible structures, if the sensors and actuators are collocated and also dual, for example, force actuators and velocity sensors or torque actuators and angular rate sensors, the plant transfer function is positive real. In practice, the prospects for controlling such systems are quite good since, if sensor and actuator dynamics and time delays are negligible, stability is unconditionally guaranteed as long as the controller is strictly positive real.[1–9]

In this paper, we consider robustness analysis of feedback systems with special attention given to flexible structure control. One of the most important types of controller architectures for this class of systems is decentralized, collocated rate feedback. As noted, in a negative feedback configuration, a controller that is strictly positive real cannot destabilize the system. Hence, such a control system will be unconditionally robust to natural frequency and damping uncertainties. In addition, the decentralized controller structure provides a high degree of fault tolerance. Hence, this is a popular controller architecture for practical flexible structure control. The simplest form of decentralized, collocated rate feedback is static diagonal controller feedback and, hence, this special case is considered first. To obtain desired system performance, however, it is often necessary to design more complex compensation. Hence, the initial results for static feedback are generalized to dynamic (nondiagonal) compensation.

Most of the performance robustness analysis results in the literature will not always yield finite performance bounds for the case of closed-loop systems consisting of uncertain positive real plants controlled by strictly positive real compensators. These results are obviously conservative since this class of systems is unconditionally stable. This paper uses majorant analysis[10–12] to develop tests that yield finite frequency-dependent performance bounds for the described case.

Majorant theory was originally developed by Dahlquist[12] to produce bounds for the solutions of systems of differential equations. The corresponding bounding techniques focus on providing upper bounds on subblocks of matrices and inverse matrices. Similar bounding procedures have been used in the work of researchers in large-scale systems analysis.[13,14] The more recent results of Refs. 15–18 apply majorant techniques to produce robust performance bounds for uncertain linear systems.

In Refs. 15–18 performance is measured basically in three ways. References 15 and 16 measure performance in terms of second-order statistics. In particular, bounds are obtained on the steady-state variances of selected system variables. In Ref. 17, performance is expressed in terms of the frequency response of selected system

Received Oct. 3, 1994; revision received Aug. 15, 1995; accepted for publication Sept. 9, 1995.

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outputs. This result led to a new upper bound for the structured singular value. Finally, Ref. 18 considers the transient response of certain system outputs, a performance measure that had not previously been treated in the robustness literature. A common feature of these results and most other robustness results, with the possible exception of methods based on extensions of Popov analysis and parameter-dependent Lyapunov functions,[19,20] is that they do not predict unconditional stability for feedback systems consisting of a positive real plant controlled by a strictly positive real controller.

This paper uses majorant analysis to develop tests for robust stability and performance that predict unconditional stability for the described case and also yield robust performance bounds. As in Refs. 17 and 21, this paper considers the frequency domain behavior of a given system. The results are specialized to the case of static, decentralized collocated diagonal rate feedback and dynamic, collocated (nondiagonal) rate feedback. The bounds developed here are illustrated with examples chosen from this class of problems and compared with the performance bound obtained in Ref. 17 and the performance bound resulting from real structured singular value analysis.[21,22] It is seen that the new bounds are much less conservative than the alternative bounds. Hence, in effect, these new bounds provide new, less conservative real-μ bounds for uncertain structural systems controlled by strictly positive real compensators.

Note that majorant bounding fundamentally differs from the standard singular value and structured singular value bounding, associated with H_∞ and μ analysis, by directly considering the multiobjective nature of the performance criteria. In particular, whereas singular value performance measures only provide bounds on the L_2 norm of the performance variables, majorant analysis provides bounds on the individual elements of the performance variables. Hence, majorant bounding is less crude than singular value bounding and is more closely related to the type of performance specifications that naturally arise in real-world engineering problems.

The paper is organized as follows. Section II presents the necessary mathematical foundation. Section III gives results relating to strictly positive real feedback of positive real vibrational systems. Section IV develops robust performance bounds for the aforementioned systems. Section V specializes the performance bounds to the case of static, decentralized collocated diagonal rate feedback. In Sec. VI we extend the results of Sec. V to dynamic, centralized output feedback. Section VII presents several illustrative examples that demonstrate the effectiveness of the proposed approach. Finally, Sec. VIII presents conclusions. Note that in the notation provided in the nomenclature the matrices and vectors are, in general, assumed to be complex.

II. Mathematical Preliminaries

In this section we establish several definitions and lemmas. A nonnegative matrix A is a matrix with nonnegative elements, i.e., $A \geq\geq 0$. For given matrix norms a block-norm matrix[10,11] of a given partitioned matrix is a nonnegative matrix, each of whose elements are the norm of the corresponding subblock. The modulus matrix of $A \in \mathbb{C}^{m\times n}$ is the nonnegative matrix

$$|A|_M \triangleq \left[\left|A_{(i,j)}\right|\right] \tag{1}$$

Note that the modulus matrix is a special case of a block-norm matrix. Subsequent analysis will use the submultiplicative relation

$$|AB|_M \leq\leq |A|_M |B|_M \tag{2}$$

for $B \in \mathbb{C}^{n\times p}$.

A majorant[12] is an element-by-element upper bound for a block-norm matrix. Specifically, $\hat{A}$ is a majorant of $A \in \mathbb{C}^{m\times n}$ if

$$|A|_M \leq\leq \hat{A} \tag{3}$$

Let $A \in \mathbb{C}^{n\times n}$. Then $\check{A} \in \mathbb{R}^{n\times n}$ is a minorant[12] of A if

$$\check{A}_{(i,i)} \leq \left|A_{(i,i)}\right| \tag{4}$$

$$\check{A}_{(i,j)} \leq -\left|A_{(i,j)}\right| \qquad i \neq j \tag{5}$$

The following lemma is a direct consequence of the preceding definitions.

Lemma 2.1. Let $Z \in \mathbb{C}^{n\times n}$. If $\check{Z}_d$ is a minorant of Z_d and $\hat{Z}_{\mathrm{od}}$ is a majorant of Z_{od}, then $\check{Z}_d - \hat{Z}_{\mathrm{od}}$ is a minorant of Z.

A matrix $Z \in \mathbb{R}^{n\times n}$ is a nonsingular M matrix[23–25] if it has nonpositive off-diagonal elements (i.e., $Z_{(i,j)} \leq 0$ for $i \neq j$) and positive principal minors. Recall that the inverse of a nonsingular M matrix is a nonnegative matrix.[23–25]

The next two lemmas are key to the development of the robust performance bounds of the following sections. The proofs of these lemmas are based on the relationship between minorants and M matrices.

Lemma 2.2. Assume $Z \in \mathbb{C}^{n\times n}$ and $\check{Z} \in \mathbb{R}^{n\times n}$ satisfy

$$\check{Z}_{(i,i)} \leq \max\left\{\mathrm{He}\, Z_{(i,i)}, \quad \left|\mathrm{Sh}\, Z_{(i,i)}\right|\right\}, \qquad i = 1, \ldots, n \tag{6}$$

$$\check{Z}_{(i,j)} \leq -\left|Z_{(i,j)}\right|, \qquad i \neq j \tag{7}$$

Then, $\check{Z}$ is a minorant of Z. Furthermore, if $\check{Z}$ is a nonsingular M matrix, then Z is nonsingular and

$$|Z^{-1}|_M \leq\leq \check{Z}^{-1} \tag{8}$$

Proof. Note that $\mathrm{He}\, Z_{(i,i)} \leq |Z_{(i,i)}|$ and $|\mathrm{Sh}\, Z_{(i,i)}| \leq |Z_{(i,i)}|$; hence, it follows from Eq. (6) that $\check{Z}_{(i,i)} \leq |Z_{(i,i)}|$. Hence, if Z satisfies Eqs. (6) and (7), then $\check{Z}$ is a minorant of Z. Finally, Eq. (8) follows from Proposition 1 of Ref. 12. □

Lemma 2.3. Let $Q \in \mathbb{C}^{n\times n}$ and let $\underline{q}$ be a positive scalar satisfying

$$\underline{q} \leq \max\{\lambda_{\min}[\mathrm{He}\, Q], \quad \lambda_{\min}[\jmath\, \mathrm{Sh}\, Q], \quad \lambda_{\min}[-\jmath\, \mathrm{Sh}\, Q]\} \tag{9}$$

Then

$$\|Q^{-1}\|_s \leq \underline{q}^{-1} \tag{10}$$

Proof. Note that for all $x \in \mathbb{C}^n, x \neq 0$, it follows from the Cauchy–Schwarz inequality that

$$\lambda_{\min}[\mathrm{He}\, Q] \leq \frac{x^*[\mathrm{He}\, Q]x}{x^*x} \leq \frac{|x^*Qx|}{x^*x} \leq \frac{\|Qx\|_2}{\|x\|_2}$$

Hence, since $\sigma_{\min}(Q) = \min_{x\in\mathbb{C}^n} \|Qx\|_2/\|x\|_2$, it follows that $\lambda_{\min}[\mathrm{He}\, Q] \leq \sigma_{\min}(Q)$. Next, since $\sigma_{\min}(\pm\jmath Q) = \sigma_{\min}(Q)$, it follows that $\lambda_{\min}[\pm\jmath\, \mathrm{Sh}\, Q] \leq \sigma_{\min}(Q)$ and, hence, using Eq. (9), $0 < \underline{q} \leq \sigma_{\min}(Q)$, which implies that Q is nonsingular. Finally, Eq. (10) is immediate by noting that $\|Q^{-1}\|_s = 1/\sigma_{\min}(Q)$. □

Lemma 2.4 (Ref. 26). Let $A, B \in \mathbb{C}^{n\times n}$. Then $\sigma_{\min}(A + B) \geq \sigma_{\min}(A) - \sigma_{\max}(B)$.

The next lemma is a direct consequence of Theorem 4.3.1 of Ref. 26.

Lemma 2.5. Let $A, B \in \mathbb{C}^{n\times n}$ be Hermitian. Then $\lambda_{\min}(A + B) \geq \lambda_{\min}(A) + \lambda_{\min}(B)$.

Finally, we establish certain key definitions used later in the paper. Let $n(s)$ and $d(s)$ be polynomials in s with real coefficients. A function $g(s)$ of the form $g(s) = n(s)/d(s)$ is called a real-rational function. The function $g(s)$ is called proper (resp., strictly proper) if $\deg n(s) \leq \deg d(s)$ [resp., $\deg n(s) < \deg d(s)$], where deg denotes the degree of the respective polynomials. A real-rational matrix function is a matrix whose elements are rational functions with real coefficients. Furthermore, a transfer function $G(s)$ is called proper (resp., strictly proper) if every element of $G(s)$ is proper (resp., strictly proper). In this paper we assume all transfer functions are real-rational matrix functions. An asymptotically stable transfer function is a transfer function each of whose poles is in the open-left half-plane. Finally, a Lyapunov stable transfer function is a transfer function each of whose poles is in the closed-left half-plane with semisimple poles on the $\jmath\omega$ axis.

A square transfer function $G(s)$ is called positive real[27] if 1) $G(s)$ is Lyapunov stable and 2) $\mathrm{He}\, G(s)$ is nonnegative definite for all $\mathrm{He}[s] > 0$. Note that if $G(s)$ is positive real and asymptotically stable, then $\mathrm{He}\, G(\jmath\omega)$ is nonnegative definite for all $\omega \in \mathbb{R}$ (Ref. 27). A square transfer function $G(s)$ is called strictly positive real[28,29] if 1) $G(s)$ is asymptotically stable and 2) $\mathrm{He}\, G(\jmath\omega)$ is positive definite for all real ω.

III. Uncertain Positive Real Plants Controlled by Strictly Positive Real Feedback

We begin by considering the following nth-order, uncertain, matrix second-order vibrational system with proportional damping and rate measurements

$$\ddot{\eta}(t) + 2Z\Omega\dot{\eta}(t) + \Omega^2\eta(t) = Bu(t) + Dw(t) \tag{11}$$

$$y(t) = C\dot{\eta}(t) \tag{12}$$

$$z(t) = E\dot{\eta}(t) \tag{13}$$

where

$$\Omega = \mathrm{diag}(\Omega_1, \ldots, \Omega_n) \tag{14}$$

$$Z = \mathrm{diag}(\zeta_1, \ldots, \zeta_n) \tag{15}$$

and where $\Omega_i > 0$, $\zeta_i > 0$, $i = 1, \ldots, n$; $u \in \mathbb{R}^{n_u}$ is the control signal; $w \in \mathbb{R}^{n_w}$ is the disturbance variable or reference signal; $y \in \mathbb{R}^{n_y}$ represents the rate measurements; and $z \in \mathbb{R}^{n_z}$ represents the performance variables that are restricted to be a linear function of the modal rates. The parameters Ω_i and ζ_i, $i = 1, \ldots, n$, denote the modal frequencies and damping ratios, respectively. It is assumed that

$$\Omega \in \boldsymbol{\Omega} \triangleq \{\Omega_0 + \Delta\Omega : |\Delta\Omega|_M \leq\leq \widehat{\Delta\Omega}\} \tag{16}$$

$$Z \in \boldsymbol{Z} \triangleq \{Z + \Delta Z : |\Delta Z|_M \leq\leq \widehat{\Delta Z}\} \tag{17}$$

$$B \in \boldsymbol{B} \triangleq \{B_0 + \Delta B : |\Delta B|_M \leq\leq \widehat{\Delta B}\} \tag{18}$$

$$C \in \boldsymbol{C} \triangleq \{C_0 + \Delta C : |\Delta C|_M \leq\leq \widehat{\Delta C}\} \tag{19}$$

$$D \in \boldsymbol{D} \triangleq \{D_0 + \Delta D : |\Delta D|_M \leq\leq \widehat{\Delta D}\} \tag{20}$$

$$E \in \boldsymbol{E} \triangleq \{E_0 + \Delta E : |\Delta E|_M \leq\leq \widehat{\Delta E}\} \tag{21}$$

where $\Omega_0 \in \mathbb{R}^{n\times n}$ is a given nominal matrix, $\Delta\Omega \in \mathbb{R}^{n\times n}$ denotes the perturbation from the nominal matrix Ω_0, and $\widehat{\Delta\Omega} \in \mathbb{R}^{n\times n}$ is the matrix majorant of $\Delta\Omega$ (and similarly for $Z \in \mathbb{R}^{n\times n}$, $B_0 \in \mathbb{R}^{n\times n_u}$, $C_0 \in \mathbb{R}^{n_y\times n}$, $D_0 \in \mathbb{R}^{n\times n_w}$, and $E_0 \in \mathbb{R}^{n_z\times n}$). Note that since Ω_0 and Z_0 are diagonal $\Delta\Omega$, $\widehat{\Delta\Omega}$, ΔZ, and $\widehat{\Delta Z}$ are assumed to be diagonal.

Next, define the ordered pairs $H_1 \triangleq (\Omega, Z)$, $H_2 \triangleq (B, C)$, $H_3 \triangleq (D, E)$, and define $\boldsymbol{H}_1, \boldsymbol{H}_2$, and $\boldsymbol{H}_3$ to be the corresponding uncertainty sets, that is, $H_1 \in \boldsymbol{H}_1 \triangleq \boldsymbol{\Omega} \times \boldsymbol{Z}$, $H_2 \in \boldsymbol{H}_2 \triangleq \boldsymbol{B} \times \boldsymbol{C}$, and $H_3 \in \boldsymbol{H}_3 \triangleq \boldsymbol{D} \times \boldsymbol{E}$. Additionally, define $H \triangleq (H_1, H_2, H_3)$ and $\boldsymbol{H} \triangleq \boldsymbol{H}_1 \times \boldsymbol{H}_2 \times \boldsymbol{H}_3$. Note that $\boldsymbol{H}_1$ is the uncertainty set corresponding to errors in the frequencies and damping ratios, whereas $\boldsymbol{H}_2$ and $\boldsymbol{H}_3$ are uncertainty sets corresponding to errors in the mode shapes. It follows from Eqs. (16–21) that $\boldsymbol{H}_1$, $\boldsymbol{H}_2$, and $\boldsymbol{H}_3$ are arcwise connected.

Furthermore, define $\theta(H, s) \triangleq \mathcal{L}[\dot{\eta}(t)]$ so that, with $\eta(0) = \dot{\eta}(0) = 0$, Eqs. (11–13) have the frequency-domain representation

$$\theta(H, s) = \Phi(H_1, s)[Bu(s) + Dw(s)] \tag{22}$$

$$y(H, s) = C\theta(H, s) \tag{23}$$

$$z(H, s) = E\theta(H, s) \tag{24}$$

where

$$\Phi(H_1, s) \triangleq \mathrm{diag}[\phi_1(s, \zeta_1, \Omega_1), \ldots, \phi_n(s, \zeta_n, \Omega_n)] \qquad H_1 \in \boldsymbol{H}_1 \tag{25}$$

and

$$\phi_i(s, \zeta_i, \Omega_i) \triangleq \frac{s}{\left(s^2 + 2\zeta_i\Omega_i s + \Omega_i^2\right)} \tag{26}$$

Note that for all $H_1 \in \boldsymbol{H}_1$, $\Phi(H_1, s)$ is positive real and

$$\mathrm{He}\,\Phi(H_1, s) > 0, \qquad H_1 \in \boldsymbol{H}_1, \qquad s \in \mathbb{C}, \qquad s \neq 0, \qquad \mathrm{He}\, s \geq 0 \tag{27}$$

Next, assume that the linear feedback law

$$u(s) = -K(s)y(s) \tag{28}$$

stabilizes the nominal system, i.e., the system (22–24) with $H_1 = (\Omega_0, Z_0)$ and $H_2 = (B_0, C_0)$. Substituting Eq. (28) into Eq. (22) gives

$$[I + \Phi(H_1, s)F(H_2, s)]\theta(H, s) = \Phi(H_1, s)Dw(s) \tag{29}$$

where

$$F(H_2, s) \triangleq BK(s)C \tag{30}$$

Now the transfer function from disturbances $w(s)$ to modal rates $\theta(H, s)$ is given by

$$\theta(H, s) = G_{w\theta}(H, s)w(s) \tag{31}$$

where $G_{w\theta}(H, s) \triangleq [I + \Phi(H_1, s)F(H_2, s)]^{-1}\Phi(H_1, s)D$. The following proposition is needed for Theorem 3.1.

Proposition 3.1. Let $H_1 \in \boldsymbol{H}_1$, $H_2 \in \boldsymbol{H}_2$, and $H \in \boldsymbol{H}$. If

$$\det[I + \Phi(H_1, s)F(H_2, s)] \neq 0, \qquad s \in \mathbb{C}, \qquad \mathrm{He}\, s \geq 0 \tag{32}$$

then $G_{w\theta}(H, s)$ is asymptotically stable.

Proof. Note that $\Phi(H_1, s)$ is asymptotically stable. Now, the result is immediate by noting that the singularities of $[I + \Phi(H_1, s)F(H_2, s)]^{-1}$ are the poles of $G_{w\theta}(H, s)$. □

For the statement of the next result, let $Q : \boldsymbol{H} \to \mathbb{C}^{n\times n}$ and define $\nu(Q)$ by

$$\nu(Q) \triangleq \max\left\{\min_{H\in\boldsymbol{H}} \lambda_{\min}[\mathrm{He}\, Q(H)],\ \min_{H\in\boldsymbol{H}} \lambda_{\min}[\jmath\,\mathrm{Sh}\, Q(H)],\ \min_{H\in\boldsymbol{H}} \lambda_{\min}[-\jmath\,\mathrm{Sh}\, Q(H)]\right\} \tag{33}$$

Theorem 3.1. If for all $H_2 \in \boldsymbol{H}_2$, $F(H_2, s)$ is positive real and asymptotically stable, then $G_{w\theta}(H, s)$ is asymptotically stable for all $H \in \boldsymbol{H}$. In addition, there exists $\underline{p}(\jmath\omega) > 0$, $\omega \in \mathbb{R}$, such that

$$\left\|\left[\Phi^{-1}(H_1, \jmath\omega) + F(H_2, \jmath\omega)\right]^{-1}\right\|_s \leq \underline{p}^{-1}(\jmath\omega) \qquad \omega \in \mathbb{R}, \qquad \omega \neq 0 \tag{34}$$

and

$$\underline{p}(\jmath\omega) \leq \nu[\Phi^{-1}(\jmath\omega) + F(\jmath\omega)] \tag{35}$$

Proof. See Appendix A. □

Remark 3.1. The norm bound (34) lays the foundation for one of the performance bounds given in the next section.

Remark 3.2. In the scalar case $\nu(Q)$ can be written as

$$\nu(Q) = \max\left\{\min_{H\in\boldsymbol{H}} \mathrm{He}\, Q(H),\ \min_{H\in\boldsymbol{H}} |\mathrm{Sh}\, Q(H)|\right\} \tag{36}$$

IV. Frequency Domain Performance Bounds

In this section we assume that $F(H_2, s)$ is positive real and asymptotically stable for all $H_2 \in \boldsymbol{H}_2$. Next, define

$$\Gamma(H_1, H_2, s) \triangleq [I + \Phi(H_1, s)F(H_2, s)]^{-1}\Phi(H_1, s) \tag{37}$$

so that Eq. (29) yields

$$\theta(H, s) = \Gamma(H_1, H_2, s)Dw(s) \tag{38}$$

Furthermore, let $\hat{\Gamma}(\jmath\omega)$ denote a majorant of $\Gamma(H_1, H_2, \jmath\omega)$ for all $H_1 \in \boldsymbol{H}_1$, $H_2 \in \boldsymbol{H}_2$, and $\omega \in \mathbb{R}$, that is,

$$|\Gamma(H_1, H_2, \jmath\omega)|_M \leq\leq \hat{\Gamma}(\jmath\omega), \qquad H_1 \in \boldsymbol{H}_1, \qquad H_2 \in \boldsymbol{H}_2 \tag{39}$$

Then, applying inequality (2) to Eq. (38) gives

$$|\theta(H, \jmath\omega)|_M \leq\leq \hat{\Gamma}(\jmath\omega)\hat{D}|w(\jmath\omega)|_M, \qquad H \in \boldsymbol{H} \tag{40}$$

Similarly, applying inequality (2) to Eq. (24) and using Eq. (40) gives

$$|z(H, \jmath\omega)|_M \leq\leq \hat{E}\hat{\theta}(\jmath\omega), \qquad H \in \boldsymbol{H} \tag{41}$$

where

$$\hat{\theta}(\jmath\omega) \triangleq \hat{\Gamma}(\jmath\omega)\hat{D}|w(\jmath\omega)|_M \tag{42}$$

Equations (41) and (42) indicate that performance bounding requires the computation of $\hat{\Gamma}(\jmath\omega)$ satisfying Eq. (39). Note that $\Gamma(H_1, H_2, 0) = 0$ for all $H_1 \in \boldsymbol{H}_1$ and $H_2 \in \boldsymbol{H}_2$ and, hence, we choose $\hat{\Gamma}(0) = 0$. Next, we present two alternative choices of $\hat{\Gamma}(\jmath\omega)$, $\omega \in \mathbb{R}$, $\omega \neq 0$. The first theorem is a direct consequence of Theorem 3.1.

Theorem 4.1. Let $\omega \in \mathbb{R}$, $\omega \neq 0$. Assume that for all $H_2 \in \boldsymbol{H}_2$, $F(H_2, s)$ is positive real and asymptotically stable. Then there exists $\underline{p}(\jmath\omega) > 0$ satisfying Eq. (35) and

$$|\Gamma(H_1, H_2, \jmath\omega)|_M \leq\leq \hat{\Gamma}_0(\jmath\omega), \qquad H_1 \in \boldsymbol{H}_1, \qquad H_2 \in \boldsymbol{H}_2 \tag{43}$$

where

$$\hat{\Gamma}_0(\jmath\omega) = \underline{p}^{-1}(\jmath\omega)U_n \tag{44}$$

and U_n denotes the $n \times n$ matrix with all unity elements.

Let $\mathbb{D}^n$ denote the set of $n \times n$ diagonal matrices, let $Q : \boldsymbol{H} \to \mathbb{D}^n$, and define the function $\nu_D(Q)$ by

$$\nu_D(Q) \triangleq \mathrm{diag}\left[\nu\left(Q_{(1,1)}\right), \ldots, \nu\left(Q_{(n,n)}\right)\right] \tag{45}$$

For the statement of the next result let $F_d(H_2, s)$ and $F_{\mathrm{od}}(H_2, s)$, respectively, denote the diagonal and off-diagonal matrices corresponding to $F(H_2, s)$. Furthermore, let $\hat{F}_{\mathrm{od}}(\jmath\omega)$ be a majorant of $F_{\mathrm{od}}(H_2, \jmath\omega)$ for all $H_2 \in \boldsymbol{H}_2$, that is,

$$\max_{H_2 \in \boldsymbol{H}_2} \left|[F_{\mathrm{od}}(H_2, \jmath\omega)]_{(i,j)}\right| \leq [\hat{F}_{\mathrm{od}}(\jmath\omega)]_{(i,j)} \tag{46}$$

Theorem 4.2. Let $\omega \in \mathbb{R}$, $\omega \neq 0$. Assume that for all $H_2 \in \boldsymbol{H}_2$, $F(H_2, s)$ is positive real and asymptotically stable. Then there exists an $n \times n$ diagonal matrix $P(\jmath\omega) \gg 0$ such that

$$P(\jmath\omega) \leq\leq \nu_D\left[\Phi^{-1}(\jmath\omega) + F_d(\jmath\omega)\right] \tag{47}$$

Furthermore, if $P(\jmath\omega) - \hat{F}_{\mathrm{od}}(\jmath\omega)$ is a nonsingular M matrix then

$$|\Gamma(H_1, H_2, \jmath\omega)|_M \leq\leq (P(\jmath\omega) - \hat{F}_{\mathrm{od}}(\jmath\omega))^{-1}, \quad H_1 \in \boldsymbol{H}_1 \quad H_2 \in \boldsymbol{H}_2 \tag{48}$$

Proof. Note that it follows from Lemma 2.2 that $P(\jmath\omega) - \hat{F}_{\mathrm{od}}(\jmath\omega)$ is a minorant of $\Phi^{-1}(H_1, \jmath\omega) + F(H_2, \jmath\omega)$ for all $\omega \in \mathbb{R}$, $\omega \neq 0$, $H_1 \in \boldsymbol{H}_1$, and $H_2 \in \boldsymbol{H}_2$. Now Eq. (48) follows from Eq. (8) of Lemma 2.2. □

Remark 4.1. Note that for the case $n = 1$ Theorems 4.1 and 4.2 yield the same performance bound. In the case $n > 1$, however, the performance bounds are obtained by computing the minimum of the bounds given by Theorems 4.1 and 4.2.

V. Performance Bounds for Static, Decentralized Collocated Rate Feedback

In this section we give frequency-dependent performance bounds for positive real systems controlled by static, decentralized collocated diagonal rate feedback. Specifically, we assume that $C = B^T$ and $K(s) = \kappa\kappa^T$, where $\kappa = \mathrm{diag}(\kappa_1, \ldots, \kappa_m)$, $\kappa_i \neq 0$. Hence,

$$F(H_2, s) = B\kappa\kappa^T B^T \tag{49}$$

We now present two theorems that are useful for computing bounds given in Theorems 4.1 and 4.2. First, however, define $\mathcal{S} : \mathbb{R} \to \mathbb{R}$ as

$$\mathcal{S}(\alpha) \triangleq \begin{cases} \alpha, & \alpha \geq 0 \\ 0, & \alpha < 0 \end{cases} \tag{50}$$

Theorem 5.1. Let $\omega \in \mathbb{R}$, $\omega \neq 0$. If $F(H_2, s)$ is given by Eq. (49), then the hypotheses of Theorems 3.1 and 4.1 are satisfied with $\underline{p}(\jmath\omega)$ given by

$$\begin{aligned} \underline{p}(\jmath\omega) = \max \Bigg\{ & \min_k 2(\zeta_{0,k} - \widehat{\Delta\zeta}_k)(\Omega_{0,k} - \widehat{\Delta\Omega}_k) \\ & + \left[\mathcal{S}\left(\sigma_{\min}(B_0\kappa) - \left(\max_i \kappa_i\right)\|\widehat{\Delta B}\|_F\right)\right]^2, \\ & \min_k (1/\omega)(\Omega_k - \widehat{\Delta\Omega}_k)^2 - \omega, \qquad \min_k \omega - (1/\omega)(\Omega_k + \widehat{\Delta\Omega}_k)^2 \Bigg\} \end{aligned} \tag{51}$$

Proof. See Appendix B. □

Next, note that with Eq. (49) $F_d(H_2, s)$ and $F_{\mathrm{od}}(H_2, s)$ are given by

$$\begin{aligned} F_d(H_2, s) &= \mathrm{diag}\left(\sum_{j=1}^m B^2_{(1,j)}\kappa_j^2, \ldots, \sum_{j=1}^m B^2_{(n,j)}\kappa_j^2\right) \\ [F_{\mathrm{od}}(H_2, s)]_{(i,j)} &= \sum_{l=1}^m B_{(i,\ell)}\kappa_\ell^2 B_{(j,\ell)} \end{aligned} \tag{52}$$

Using Eq. (52) we obtain the following result.

Theorem 5.2. Let $\omega \in \mathbb{R}$, $\omega \neq 0$, and let $F(H_2, s)$ be given by Eq. (49). Then the hypothesis of Theorem 4.2 is satisfied with $P(\jmath\omega)$ given by

$$P(\jmath\omega) = \mathrm{diag}[P_1(\jmath\omega), \ldots, P_n(\jmath\omega)] \tag{53}$$

where

$$\begin{aligned} P_k(\jmath\omega) = \max \Bigg\{ & 2(\zeta_{0,k} - \widehat{\Delta\zeta}_k)(\Omega_{0,k} - \widehat{\Delta\Omega}_k) + \sum_{j=1}^m \Big[\mathcal{S}\Big(\big(B_{(k,j)} \\ & - \widehat{\Delta B}_{(k,j)}\big)\kappa_j\Big)\Big]^2, \qquad \min_{\Omega \in \boldsymbol{\Omega}} \left|\frac{1}{\omega}\Omega_k^2 - \omega\right| \Bigg\} \end{aligned} \tag{54}$$

In addition, the (i, j) element of $\hat{F}_{\mathrm{od}}(\jmath\omega)$ satisfying Eq. (46) of Theorem 4.2 is given by

$$[\hat{F}_{\mathrm{od}}(\jmath\omega)]_{(i,j)} = \sum_{\ell=1}^m \left(|B_{0(i,\ell)}| + \widehat{\Delta B}_{(i,\ell)}\right)\kappa_\ell^2\left(|B_{0(j,\ell)}| + \widehat{\Delta B}_{(j,\ell)}\right) \tag{55}$$

Proof. The proof is similar to that of Theorem 5.1 and hence is omitted. □

VI. Extensions to Dynamic Compensation

In this section we generalize the results of Sec. V to dynamic compensation. Once again we assume collocated rate feedback so that $C = B^T$ holds. Unlike the results of Sec. V, however, we do not assume a diagonal structure for the dynamic compensator. The following two theorems provide frequency-dependent performance bounds for positive real systems controlled by strictly positive real dynamic compensators.

Theorem 6.1. Let $\omega \in \mathbb{R}$, $\omega \neq 0$, and let $F(H_2, s) = BK(s)B^T$ where $K(s)$ is a given strictly positive real compensator. Then the hypotheses of Theorems 3.1 and 4.1 are satisfied with $\underline{p}(\jmath\omega)$ given by

$$\begin{aligned} \underline{p}(\jmath\omega) = \max \Bigg\{ & \min_k 2(\zeta_{0,k} - \widehat{\Delta\zeta}_k)(\Omega_{0,k} - \widehat{\Delta\Omega}_k) \\ & + \tfrac{1}{2}[\mathcal{S}(\sigma_{\min}(B_0M(\jmath\omega)) - \sigma_{\max}(M(\jmath\omega))\|\widehat{\Delta B}\|_F)]^2, \\ & \min_k \left[1/\omega(\Omega_{0,k} - \widehat{\Delta\Omega}_k)^2 - \omega\right] \\ & - \sigma_{\max}[\mathrm{Sh}\, K(\jmath\omega)](\sigma_{\max}(B_0) + \|\widehat{\Delta B}\|_F)^2, \\ & \min_k \left[\omega - (1/\omega)(\Omega_{0,k} + \widehat{\Delta\Omega}_k)^2\right] \\ & - \sigma_{\max}[\mathrm{Sh}\, K(\jmath\omega)][\sigma_{\max}(B_0) + \|\widehat{\Delta B}\|_F]^2 \Bigg\} \end{aligned} \tag{56}$$

where

$$2\mathrm{He}\, K(\jmath\omega) = M(\jmath\omega)M^*(\jmath\omega) \tag{57}$$

Proof. See Appendix C. □

The next theorem gives an alternative bound for the dynamic compensator case. Using Theorem 4.2 and a procedure similar to the one employed in the proof of Theorem 6.1, the following result is immediate.

Theorem 6.2. Let $\omega \in \mathbb{R}$, $\omega \neq 0$, and let $F(H_2, s) = BK(s)B^T$, where $K(s)$ is a given strictly positive real compensator. Then the hypothesis of Theorem 4.2 is satisfied with $P(\jmath\omega)$ given by

$$P(\jmath\omega) = \mathrm{diag}\,[P_1(\jmath\omega), \ldots, P_n(\jmath\omega)] \tag{58}$$

where

$$P_k(\jmath\omega) = \max\Bigg\{2(\zeta_{0,k} - \widehat{\Delta\zeta}_k)(\Omega_{0,k} - \widehat{\Delta\Omega}_k) + \lambda_{\min}[\mathrm{He}\,K(\jmath\omega)]\sum_{l=1}^{m}\left[\mathcal{S}\left(B_{0(k,l)} - \widehat{\Delta B}_{(k,l)}\right)\right]^2,$$
$$\min_{\Omega\in\mathbf{\Omega}}\left|\frac{\Omega_{0,k}^2}{\omega} - \omega\right| - \sigma_{\max}[\mathrm{Sh}\,K(\jmath\omega)] \times \sum_{l=1}^{m}\left[\left|B_{0(k,l)}\right| + \widehat{\Delta B}_{(k,l)}\right]^2\Bigg\} \tag{59}$$

In addition, $\hat{F}_{\mathrm{od}}(\jmath\omega)$ satisfying Eq. (46) is given by

$$[\hat{F}_{\mathrm{od}}(\jmath\omega)]_{(i,j)} = \sigma_{\max}[K(\jmath\omega)]\left[\sum_{k=1}^{m}\left(\left|B_{0(i,k)}\right| + \widehat{\Delta B}_{(i,k)}\right)^2\right]^{\frac{1}{2}} \times \left[\sum_{k=1}^{m}\left(\left|B_{0(j,k)}\right| + \widehat{\Delta B}_{(j,k)}\right)^2\right]^{\frac{1}{2}} \tag{60}$$

Remark 6.1. Note that for the case where a positive real system is controlled by a diagonal compensator the performance bounds given by Theorems 6.1 and 6.2 will, in general, be more conservative than the bounds given by Theorems 5.1 and 5.2, because the performance bounds given by Theorems 6.1 and 6.2 are derived for a general dynamic controller architecture as opposed to the performance bounds given by Theorems 5.1 and 5.2 that explicitly exploit the diagonal controller structure.

Remark 6.2. Note that the computation of the performance bounds given by Theorems 5.1, 5.2, 6.1, and 6.2 involve simple computations of $\underline{p}(\jmath\omega)$ and $P(\jmath\omega)$ at discrete frequency points given by closed-form expressions involving known problem data. This is in contrast to the structured singular value bounds[22,30] that lead to a multidimensional optimization over $\boldsymbol{H}_1 \times \boldsymbol{H}_2$ that although convex is considerably more computationally demanding in comparison.

Finally, since $\underline{p}(\jmath\omega)$ and $P(\jmath\omega)$ at discrete frequency points in the relevant frequency interval are required to compute the frequency-dependent performance bounds, we present an algorithm needed for obtaining these bounds for a specified uncertain system. Let $[\underline{\omega}, \bar{\omega}]$ denote the frequency interval for which robust performance bounds are required.

Algorithm 6.1. To solve for the robust performance bounds carry out the following steps.

Step 1. Discretize the frequency interval $[\underline{\omega}, \bar{\omega}]$ (linearly or logarithmically) and let $v \in \mathbb{R}^k$ be the vector of discrete points in $[\underline{\omega}, \bar{\omega}]$ with $v_1 = \underline{\omega}$ and $v_k = \bar{\omega}$.

Step 2. Initialize $l = 1$ and $\omega = v_l$.

Step 3. Compute $\underline{p}(\jmath\omega)$ using Theorem 5.1 (or Theorem 6.1).

Step 4. Compute $P(\jmath\omega)$ using Theorem 5.2 (or Theorem 6.2) and $\hat{F}_{\mathrm{od}}(\jmath\omega)$ and let $\Pi(\jmath\omega) \triangleq P(\jmath\omega) - \hat{F}_{\mathrm{od}}(\jmath\omega)$.

Step 5. If $\Pi(\jmath\omega)$ is a nonsingular M matrix, then let $\hat{z}(\jmath\omega)$ be such that $[\hat{z}(\jmath\omega)]_{(i)} = \min\{[\underline{p}(\jmath\omega)\hat{E}U_n\hat{D}|w(\jmath\omega)|_M]_{(i)}, [\hat{E}\Pi^{-1}(\jmath\omega)\hat{D}|w(\jmath\omega)|_M]_{(i)}\}$, otherwise let $\hat{z}(\jmath\omega) = \underline{p}(\jmath\omega)\hat{E}U_n\hat{D}|w(\jmath\omega)|_M$, where $U_n \in \mathbb{R}^{n\times n}$ is the matrix with all unity elements.

Step 6. Repeat steps 3–7 for $l = 2, \ldots, k$.

VII. Illustrative Numerical Examples

In this section we present several illustrative numerical examples that demonstrate the effectiveness of the proposed frequency domain majorant bounds. We compare the present bounds to the complex block-structured majorant bound of Ref. 17 and the real structured singular value bound given in Ref. 22. To provide a frequency-by-frequency comparison of these bounds we generalize the notion of real structured singular value bounds to performance bounds in terms of Euclidean vector norms of the system output over frequency. Specifically, for the real-μ performance bounds we compute

$$\|z(\jmath\omega)\|_2 \leq \mu_{\mathbf{\Delta}}(G(\jmath\omega))\|w(\jmath\omega)\|_2 \tag{61}$$

where $\mu_{\mathbf{\Delta}}(G(\jmath\omega))$ refers to the real-μ bound[22] with respect to the block-structured uncertainty $\mathbf{\Delta}$ that includes a performance block and where $G(s)$ is a block-partitioned transfer function arising in the robust H_∞ standard problem. For details see Ref. 30. Note that for multi-input/multi-output systems it follows from Eq. (61) that $|z(\jmath\omega)|_M \leq\leq \mu_{\mathbf{\Delta}}(G(\jmath\omega))U_n|w(\jmath\omega)|_M$, where $U_n \in \mathbb{R}^{n\times n}$ is the matrix with all unity elements, so that the frequency-dependent majorant bounds developed in the paper and the structured singular value bounds obtained via Eq. (61) can be made commensurate for comparison purposes.

Example 7.1 ($n = 1$, damping uncertainty). Our first example considers performance bounding for the case $n = 1$ with one control input and damping uncertainty. The closed-loop system is given by Eq. (29) with

$$\Phi^{-1}(H_1, s) = \frac{s^2 + 2\zeta\Omega s + \Omega^2}{s}, \qquad F(H_2, s) = k_1 b^2$$
$$k_1 = 2, \qquad D = b = 1$$

and with frequency in radian per second

$$\Omega = 10(2\pi), \qquad \zeta = 0.01, \qquad \widehat{\Delta\zeta} = 0.009$$

In this case, Theorems 4.1 and 4.2 give the same performance bound, which is shown in Fig. 1. For this example, the complex block-structured majorant bound[17] is nonconservative, whereas the real structured singular value bound gives the most conservative performance predictions in terms of peak upper bounds.

Example 7.2 ($n = 1$, frequency uncertainty). This example considers the same case as example 7.1 except that the damping ratio

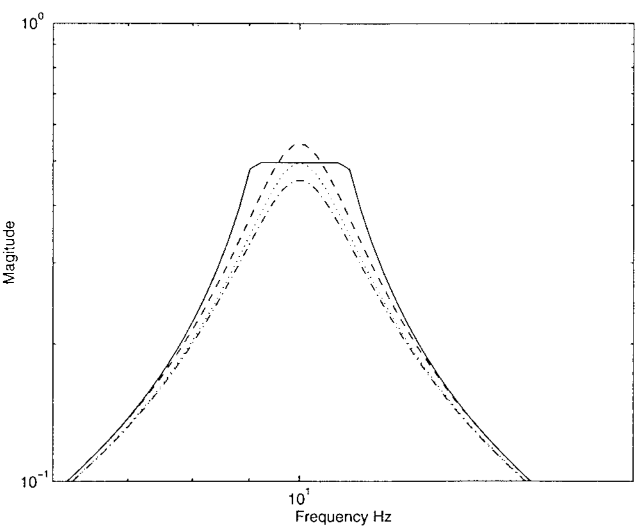

Fig. 1 Performance bounds for example 7.1, $n = 1$, damping uncertainty: ——, Theorems 5.1 and 5.2; -·-·-, nominal; ·····, perturbed, Ref. 17; and – –, Ref. 22.

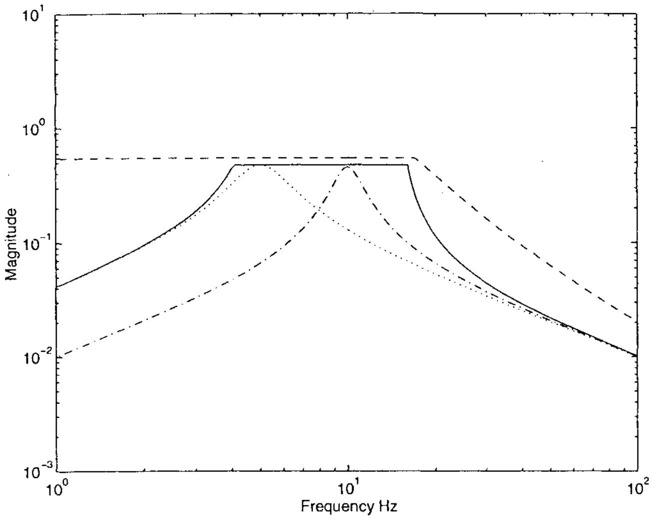

Fig. 2 Performance bounds for example 7.2, n = 1, frequency uncertainty: ——, Theorems 5.1 and 5.2; -·-·, nominal; ····, perturbed; and – –, Ref. 22.

is constant, whereas the frequency in radian per second is uncertain with

$$\widehat{\Delta\Omega} = 5(2\pi)$$

For the assumed uncertainty range the complex block-structured majorant bound[17] is infinite since, in this case, the method predicts instability. The performance bound obtained via Theorems 5.1 and 5.2 give a tight finite performance bound, whereas the real structured singular value bound gives a conservative performance bound. This is shown in Fig. 2.

Example 7.3 (n = 3, frequency uncertainty in the first mode). This example considers performance bounds for three modes with one control input and frequency uncertainty in the first mode. The closed-loop system is given by Eq. (29) with

$$\Phi^{-1}(H_1, s) = \mathrm{diag}\left[\frac{s^2 + 2\zeta_1\Omega_1 s + \Omega_1^2}{s}, \ldots, \frac{s^2 + 2\zeta_3\Omega_3 s + \Omega_3^2}{s}\right]$$

$$F(H_2, s) = k_1 B B^T, \qquad D = B, \qquad k_1 = 2, \qquad B = [1, 1, 1]^T$$

and with frequency in radian per second

$$\{\Omega_1, \Omega_2, \Omega_3\} = \{10(2\pi), 50(2\pi), 100(2\pi)\}$$

$$\{\zeta_1, \zeta_2, \zeta_3\} = \{0.005, 0.01, 0.0025\}$$

$$\{\widehat{\Delta\Omega}_1, \widehat{\Delta\Omega}_2, \widehat{\Delta\Omega}_3\} = \{5(2\pi), 0, 0\}$$

Once again the complex block-structured majorant bound[17] gives infinite performance predictions. The performance bound obtained via Theorems 5.1 and 5.2 shown in Fig. 3 gives a finite performance bound. This bound was obtained by computing the minimum of the performance bounds given by Theorems 5.1 and 5.2 for each frequency. The real structured singular value bound gives a finite but more conservative performance bound.

Example 7.4 (n = 3, frequency and mode shape uncertainty). To compare the performance bounds obtained by Theorems 5.1 and 5.2 and Theorems 6.1 and 6.2 this example considers the same case as example 7.3 with two sensors and two actuators with both frequency and mode shape uncertainty. Specifically, the frequency uncertainty is as in example 7.3 and the mode shape uncertainty with the assumed static diagonal controller are

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \widehat{\Delta B} = \begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 0.01 \\ 0.01 & 0.01 \end{bmatrix}, \qquad K = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

The corresponding bounds are shown in Fig. 4. As noted in Remark 6.1, since the bounds obtained by Theorems 6.1 and 6.2 are for general nondiagonal positive real compensators they do not exploit, in this case, the diagonal structure of the compensator and, hence, are more conservative than the bounds obtained by Theorems 5.1 and 5.2.

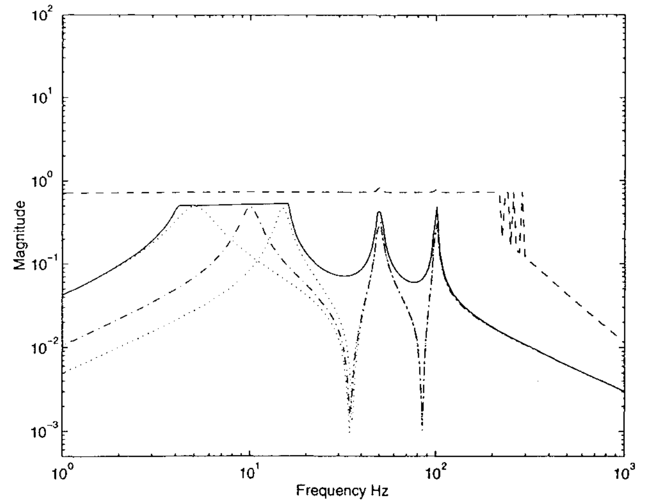

Fig. 3 Performance bounds for example 7.3, n = 3, frequency uncertainty: ——, Theorems 5.1 and 5.2; -·-·, nominal; ····, perturbed; and – –, Ref. 22.

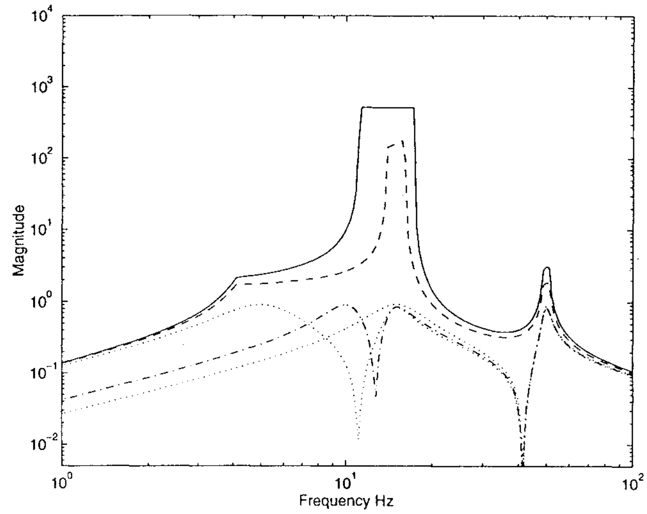

Fig. 4 Performance bounds from Theorems 5.1 and 5.2 and Theorems 6.1 and 6.2: ——, Theorems 6.1 and 6.2; – –, Theorems 5.1 and 5.2; -·-·, nominal; and ····, perturbed.

Next, using the frequency domain performance bounds given by Theorems 6.1 and 6.2 along with the controller synthesis framework presented in Ref. 8 (see also Ref. 9) for constructing strictly positive real dynamic compensators, we apply our results to a simply supported Euler–Bernoulli beam with multiple-frequency uncertainty.

Example 7.5 (n = 5, Euler–Bernoulli beam). Consider the simply supported Euler–Bernoulli beam with governing partial differential equation for the transverse deflection $w(x, t)$ given by

$$m(x)\frac{\partial^2 w(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right] = f(x,t) \tag{62}$$

and with boundary conditions

$$w(x,t)|_{x=0,L} = 0, \qquad EI\frac{\partial^2 w(x,t)}{\partial x^2}\bigg|_{x=0,L} = 0 \tag{63}$$

where $m(x)$ is mass per unit length and $EI(x)$ the flexural rigidity with E Young's modulus of elasticity and $I(x)$ the cross-sectional area moment of inertia about an axis normal to the plane of vibration and passing through the center of the cross-sectional area. Finally,

$f(x, t)$ is the force distribution resulting from control actuation and external disturbances. Assuming uniform beam properties, the modal decomposition of this system has the form

$$w(x, t) = \sum_{r=1}^{\infty} W_r(x) q_r(t) \tag{64}$$

$$\int_0^L m W_r^2(x)\,\mathrm{d}x = 1, \qquad W_r(x) = \sqrt{\frac{2}{mL}} \sin\frac{r\pi x}{L}$$

$$r = 1, 2, \ldots \tag{65}$$

where, assuming uniform proportional damping, the modal coordinates q_r satisfy

$$\ddot{q}_r(t) + 2\zeta\Omega_r \dot{q}_r(t) + \Omega_r^2 q_r(t) = \int_0^L f(x, t) W_r(x)\,\mathrm{d}x$$

$$r = 1, 2, \ldots \tag{66}$$

For simplicity assume $L = \pi$ and $m = EI = 2/\pi$ so that $\sqrt{(2/mL)} = 1$. Furthermore, we place a collocated velocity/force actuator pair at $x = 0.55L$. Finally, modeling the first five modes, defining the plant state as $x = [q_1, \dot{q}_1, \ldots, q_5, \dot{q}_5]^T$, and defining the performance of the beam in terms of the velocity at $x = 0.7L$, the resulting state–space model and problem data are

$$\dot{x}(t) = \hat{A}x(t) + \hat{B}u(t) + D_1 w(t)$$

$$y(t) = \hat{C}x(t) + D_2 w(t)$$

where

$$\hat{A} = \underset{i=1,\ldots,5}{\text{block-diag}} \begin{bmatrix} 0 & 1 \\ -\Omega_i^2 & -2\zeta\Omega_i \end{bmatrix}, \qquad \Omega_i = i^2, \qquad \zeta = 0.01$$

$$\hat{B} = \hat{C}^T$$

$$= [0 \;\; 0.9877 \;\; -0.309 \;\; 0 \;\; -0.891 \;\; 0 \;\; 0.5878 \;\; 0 \;\; 0.7071]^T$$

$$D_1 = [\hat{B} \;\; 0_{10\times 1}], \qquad D_2 = [0 \;\; 1.9]$$

with performance variables

$$z(t) = E_1 x(t) + E_2 u(t)$$

where

$$E_1 = \begin{bmatrix} 0 & 0.809 & 0 & -0.951 & 0 & 0.309 & 0 & 0.5878 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_2 = [0 \;\; 1.9]^T$$

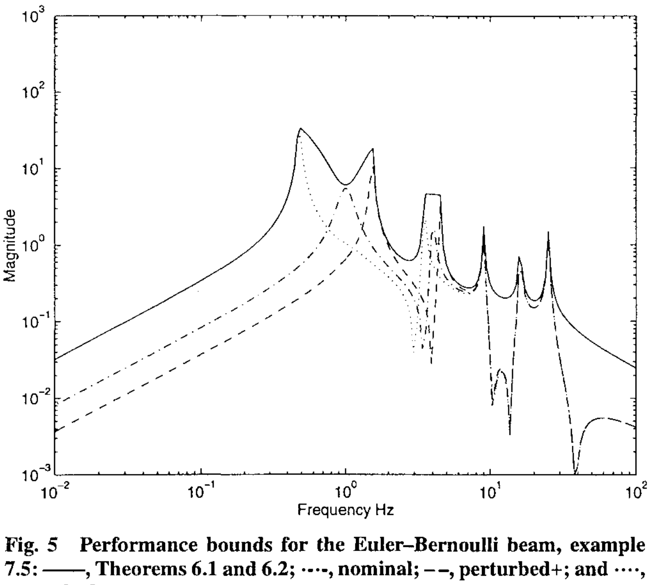

Fig. 5 Performance bounds for the Euler–Bernoulli beam, example 7.5: ——, Theorems 6.1 and 6.2; -·-·, nominal; – –, perturbed+; and ····, perturbed−.

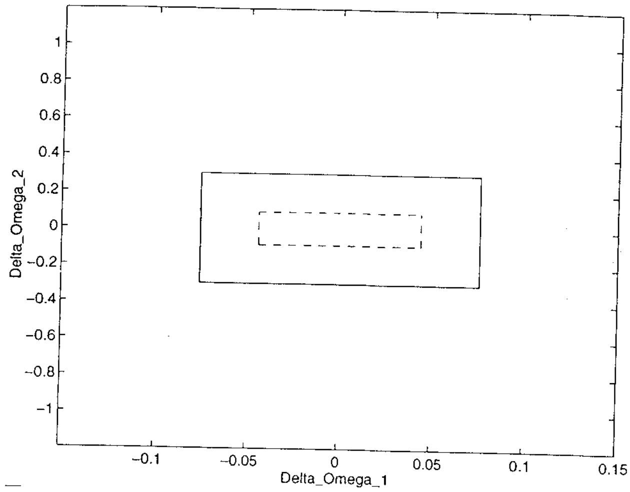

Fig. 6 Guaranteed stability predictions for example 7.5: unconditional, Theorems 6.1 and 6.2; ——, Ref. 22; and – –, Ref. 17.

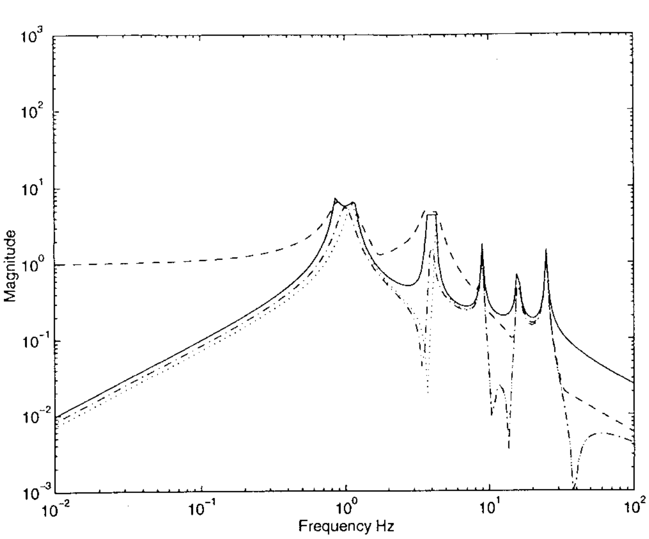

Fig. 7 Comparison of performance bounds for example 7.5: ——, Theorems 6.1 and 6.2; -·-·, nominal; ····, perturbed; and – –, Ref. 22.

Using the approach of Ref. 8 (see also Theorem 3.2 of Ref. 9) we design a strictly positive real dynamic compensator $K(s)$. Next, we assume frequency uncertainty in both Ω_1 and Ω_2 with $\widehat{\Delta\Omega_1} = 0.5$ and $\widehat{\Delta\Omega_2} = 0.5$. The corresponding performance bound obtained via Theorems 6.1 and 6.2 is given in Fig. 5. Once again this bound was obtained by computing the minimum of the performance bounds given by Theorems 6.1 and 6.2 for each frequency. The complex block-structured majorant bound[12] and the real structured singular value bound predict instability for the assumed uncertainty range and, hence, give infinite performance bounds. To compare the stability predictions using all three methods, the maximum uncertainty range in the natural frequencies for which the three methods guarantee stability is found. Specifically, complex block-structured majorant analysis[17] predicts stability for the range of $\widehat{\Delta\Omega_1} < 0.044$ and $\widehat{\Delta\Omega_2} < 0.09$, whereas the real structured singular value analysis predicts stability for the range of $\widehat{\Delta\Omega_1} < 0.076$ and $\widehat{\Delta\Omega_2} < 0.3$. The parameter space for these predictions is shown in Fig. 6. Note that the positive real result guarantees unconditional stability. Next, to compare the performance bound obtained via Theorems 6.1 and 6.2 and the real structured singular value bound, we choose $\widehat{\Delta\Omega_1} = 0.076$ and $\widehat{\Delta\Omega_2} = 0.3$ that correspond to the uncertainty range for which both these methods guarantee stability. The comparison is shown in Fig. 7. Note that the proposed frequency domain majorant bound gives a tight performance bound over the frequency range corresponding to the modal frequency uncertainty in comparison with the real structured singular value bound (real-μ performance bound). Unlike the trend exhibited in the previous examples, however, at higher frequencies corresponding to nominal modal frequency range the performance bound for the real structured singular value bound appears to be less conservative than the majorant bound.

VIII. Conclusion

This paper developed frequency domain performance bounds for closed-loop systems consisting of positive real plants and strictly positive real compensators. The results are developed by using certain properties of majorant analysis. Unlike previous results in robustness analysis, the performance bounds remain finite even when the uncertainty is made arbitrarily large. The examples compared the new bounds with a previous majorant bound and the corresponding mixed-μ bound from structured singular value analysis. In all cases the new bounds was much less conservative than the alternative bounds. Future work will involve extending these results to reduce the conservatism in the analysis of closed-loop systems for which the plant and controller are positive real only over a particular frequency band. This will allow the consideration of more realistic vibrational models that account for sensor and actuator dynamics along with time delays.

Appendix A: Proof of Theorem 3.1

Let $H_1 \in \boldsymbol{H}_1$ and $H_2 \in \boldsymbol{H}_2$, let $s \in \mathbb{C}, s \neq 0$, He $s \geq 0$, and let $x \in \mathbb{C}^n, x \neq 0$, and $\lambda \in \mathbb{C}$ be such that $\Phi(H_1, s)x = \lambda x$, and hence, $x^*\Phi^*(H_1, s) = \lambda^* x^*$. Then, it follows from Eq. (27) that

$$x^*[\mathrm{He}\,\Phi(H_1, s)]x = \mathrm{He}\,\lambda x^* x > 0$$

and, hence, $\det \Phi(H_1, s) \neq 0$. In addition,

$$\mathrm{He}\,\Phi^{-1}(H_1, s) = \Phi^{-1}(H_1, s)[\mathrm{He}\,\Phi(H_1, s)]\Phi^{-*}(H_1, s) > 0$$

Since, $F(H_2, s)$ is positive real and asymptotically stable it follows that He $F(H_2, s) \geq 0$ and, hence,

$$\mathrm{He}\left[\Phi^{-1}(H_1, s) + F(H_2, s)\right] > 0 \tag{A1}$$

which implies $\det[\Phi^{-1}(H_1, s) + F(H_2, s)] \neq 0$. Furthermore, $\det[I + \Phi(H_1, s)F(H_2, s)] = \det \Phi^{-1}(H_1, s) \det[\Phi^{-1}(H_1, s) + F(H_2, s)] \neq 0$. Next, note that $\Phi(H_1, 0) = 0$, which implies that $\det[I + \Phi(H_1, s)F(H_2, s)] = 1$ for $s = 0$ and, hence, Eq. (32) is satisfied. Now, asymptotic stability of $G_{w\theta}(H, s)$ is a direct consequence of Proposition 3.1.

Finally, it follows from Eq. (A1) that $\nu[\Phi^{-1}(\jmath\omega) + F(\jmath\omega)]$ is positive for all $\omega \in \mathbb{R}, \omega \neq 0$ and, hence, there exists $p(\jmath\omega)$ such that Eq. (35) is satisfied. Equation (34) now follows from Lemma 2.3. □

Appendix B: Proof of Theorem 5.1

Using Lemma 2.5 it follows that

$$\lambda_{\min}\left\{\mathrm{He}\left[\Phi^{-1}(H_1, \jmath\omega) + F(H_2, \jmath\omega)\right]\right\}$$
$$\geq \lambda_{\min}\left[\mathrm{He}\,\Phi^{-1}(H_1, \jmath\omega)\right] + \lambda_{\min}[\mathrm{He}\,F(H_2, \jmath\omega)] \tag{B1}$$

Now,

$$\mathrm{He}\,\Phi^{-1}(H_1, \jmath\omega) = \mathrm{diag}(2\zeta_1\Omega_1, \ldots, 2\zeta_n\Omega_n)$$
$$= \mathrm{diag}[2(\zeta_{0,1} + \Delta\zeta_{0,1})(\Omega_{0,1} + \Delta\Omega_{0,1}), \ldots,$$
$$2(\zeta_{0,n} + \Delta\zeta_{0,n})(\Omega_{0,n} + \Delta\Omega_{0,n})] \tag{B2}$$

and, hence,

$$\min_{H_1 \in \boldsymbol{H}_1} \lambda_{\min}\left[\mathrm{He}\,\Phi^{-1}(H_1, \jmath\omega)\right] = \min_k 2(\zeta_{0,k} - \widehat{\Delta\zeta}_k)(\Omega_{0,k} - \widehat{\Delta\Omega}_k) \tag{B3}$$

Furthermore,

$$\mathrm{He}\,F(H_2, \jmath\omega) = B\kappa\kappa^T B^T \tag{B4}$$

and

$$\lambda_{\min}(B\kappa\kappa^T B^T) = \sigma_{\min}^2(B\kappa) = \sigma_{\min}^2[(B_0 + \Delta B)\kappa] \tag{B5}$$

Now, it follows from Lemma 2.4 that

$$\sigma_{\min}[(B_0 + \Delta B)\kappa] \geq \sigma_{\min}(B_0\kappa) - \sigma_{\max}(\Delta B\kappa)$$
$$\geq \sigma_{\min}(B_0\kappa) - \left(\max_i \kappa_i\right)\sigma_{\max}(\Delta B)$$
$$\geq \sigma_{\min}(B_0\kappa) - \left(\max_i \kappa_i\right)\|\widehat{\Delta B}\|_F \tag{B6}$$

Since $\sigma_{\min}[(B_0 + \Delta B)\kappa] > 0$, Eqs. (B5), (B6), and (50) yield

$$\min_{H_2 \in \boldsymbol{H}_2} \lambda_{\min}[\mathrm{He}\,F(H_2, \jmath\omega)]$$
$$\geq \left[\mathcal{S}\left(\sigma_{\min}(B_0\kappa) - \left(\max_i \kappa_i\right)\|\widehat{\Delta B}\|_F\right)\right]^2 \tag{B7}$$

It now follows from Eqs. (50), (B2), and (B6) that

$$\min_{H \in \boldsymbol{H}} \lambda_{\min}\left\{\mathrm{He}\left[\Phi^{-1}(H_1, \jmath\omega) + F(H_2, \jmath\omega)\right]\right\}$$
$$\geq \min_k 2(\zeta_{0,k} - \widehat{\Delta\zeta}_k)(\Omega_{0,k} - \widehat{\Delta\Omega}_k)$$
$$+\left[\mathcal{S}\left(\sigma_{\min}(B_0\kappa) - \left(\max_i \kappa_i\right)\|\widehat{\Delta B}\|_F\right)\right]^2 \tag{B8}$$

Once again using Lemma 2.5, it follows that

$$\lambda_{\min}\left\{\jmath\,\mathrm{Sh}\left[\Phi^{-1}(H_1, \jmath\omega) + F(H_2, \jmath\omega)\right]\right\}$$
$$\geq \lambda_{\min}\left[\jmath\,\mathrm{Sh}\,\Phi^{-1}(H_1, \jmath\omega)\right] + \lambda_{\min}\left[\jmath\,\mathrm{Sh}\,F(H_2, \jmath\omega)\right] \tag{B9}$$

Now, noting

$$\jmath\,\mathrm{Sh}\,\Phi^{-1}(H_1, \jmath\omega) = \left[\mathrm{diag}\,(1/\omega)\Omega_1^2 - \omega, \ldots, (1/\omega)\Omega_n^2 - \omega\right] \tag{B10}$$

we obtain

$$\min_{H_1 \in \boldsymbol{H}_1} \lambda_{\min}\left[\jmath\,\mathrm{Sh}\,\Phi^{-1}(H_1, \jmath\omega)\right] = \min_k (1/\omega)(\Omega_k - \widehat{\Delta\Omega}_k)^2 - \omega \tag{B11}$$

Furthermore, noting that $\jmath[F(H_2, \jmath\omega) - F^*(H_2, \jmath\omega)] = 0$ yields

$$\min_{H_2 \in \boldsymbol{H}_2} \lambda_{\min}[\jmath\,\mathrm{Sh}\,F(H_2, \jmath\omega)] = 0 \tag{B12}$$

It now follows from Eqs. (B9), (B11), and (B12) that

$$\min_{H \in \boldsymbol{H}} \lambda_{\min}\left[\jmath\,\mathrm{Sh}\,\Phi^{-1}(H_1, \jmath\omega) + \jmath\,\mathrm{Sh}\,F(H_2, \jmath\omega)\right]$$
$$\geq \min_k (1/\omega)(\Omega_k - \widehat{\Delta\Omega}_k)^2 - \omega \tag{B13}$$

Similarly,

$$\min_{H \in \boldsymbol{H}} \lambda_{\min}[-\jmath\,\mathrm{Sh}\,\Phi(H_1, \jmath\omega) - \jmath\,\mathrm{Sh}\,F(H_2, \jmath\omega)] \geq \min_k \omega$$
$$- (1/\omega)(\Omega_k + \widehat{\Delta\Omega}_k)^2 \tag{B14}$$

Equation (51) is now immediate using Eq. (34) with Eqs. (B8), (B13), and (B14). □

Appendix C: Proof of Theorem 6.1

From Eqs. (B1) and (B3) we have

$$\min_{H \in \boldsymbol{H}} \lambda_{\min}\left\{\mathrm{He}\left[\Phi^{-1}(H_1, \jmath\omega) + F(H_2, \jmath\omega)\right]\right\}$$
$$\geq \min_k 2(\zeta_{0,k} - \widehat{\Delta\zeta}_k)(\Omega_{0,k} - \widehat{\Delta\Omega}_k)$$
$$+ \min_{H_2 \in \boldsymbol{H}_2} \lambda_{\min}[\mathrm{He}\,F(H_2, \jmath\omega)] \tag{C1}$$

Next, note that

$$\lambda_{\min}[\mathrm{He}\,F(H_2, \jmath\omega)] = \lambda_{\min}[B(\mathrm{He}\,K(\jmath\omega))B^T]$$
$$= \tfrac{1}{2}\lambda_{\min}[BM(\jmath\omega)M^*(\jmath\omega)B^T]$$
$$= \tfrac{1}{2}\sigma_{\min}^2[BM(\jmath\omega)]$$
$$= \tfrac{1}{2}\sigma_{\min}^2[(B_0 + \Delta B)M(\jmath\omega)] \tag{C2}$$

Now, it follows from Lemma 2.4 that

$$\begin{aligned}\sigma_{\min}[(B_0+\Delta B)M(\jmath\omega)] &\geq \sigma_{\min}[B_0M(\jmath\omega)]-\sigma_{\max}[\Delta BM(\jmath\omega)]\\ &\geq \sigma_{\min}[B_0M(\jmath\omega)]-\sigma_{\max}[M(\jmath\omega)]\sigma_{\max}(\Delta B)\\ &\geq \sigma_{\min}[B_0M(\jmath\omega)]-\sigma_{\max}[M(\jmath\omega)]\|\widehat{\Delta B}\|_F \qquad \text{(C3)}\end{aligned}$$

Noting that $\sigma_{\min}[BM(\jmath\omega)]\geq 0$, Eqs. (C2), (C3), and (50) yield

$$\min_{H_2\in\boldsymbol{H}_2}\lambda_{\min}[\text{He}\,F(H_2,\jmath\omega)] \geq \tfrac{1}{2}[\mathcal{S}(\sigma_{\min}(B_0M(\jmath\omega))-\sigma_{\max}(M(\jmath\omega))\|\widehat{\Delta B}\|_F)]^2 \qquad \text{(C4)}$$

It now follows from Eqs. (C1) and (C4) that

$$\begin{aligned}\min_{H\in\boldsymbol{H}}\lambda_{\min}\left\{\text{He}\left[\Phi^{-1}(H_1,\jmath\omega)+F(H_2,\jmath\omega)\right]\right\} &\geq \min_k 2(\zeta_{0,k}-\widehat{\Delta\zeta}_k)(\Omega_{0,k}-\widehat{\Delta\Omega}_k)\\ &+\tfrac{1}{2}\{\mathcal{S}[\sigma_{\min}(B_0M(\jmath\omega))-\sigma_{\max}(M(\jmath\omega))\|\widehat{\Delta B}\|_F]\}^2 \qquad \text{(C5)}\end{aligned}$$

Similarly, using Eqs. (B9) and (B11) we obtain

$$\begin{aligned}\min_{H\in\boldsymbol{H}}\lambda_{\min}\left[\jmath\,\text{Sh}\,\Phi^{-1}(H_1,\jmath\omega)+\jmath\,\text{Sh}\,F(H_2,\jmath\omega)\right] &\geq \min_k(1/\omega)(\Omega_{0,k}-\widehat{\Delta\Omega}_k)^2-\omega\\ &+\min_{H_2\in\boldsymbol{H}_2}\lambda_{\min}[\jmath\,\text{Sh}\,F(H_2,\jmath\omega)] \qquad \text{(C6)}\end{aligned}$$

Hence,

$$\begin{aligned}\lambda_{\min}[\jmath\,\text{Sh}\,F(H_2,\jmath\omega)] &\geq -\sigma_{\max}[\text{Sh}\,F(H_2,\jmath\omega)]\\ &= -\sigma_{\max}[B(\text{Sh}\,K(\jmath\omega))B^T]\\ &\geq -\sigma^2_{\max}(B)\sigma_{\max}[\text{Sh}\,K(\jmath\omega)]\\ &\geq -[\sigma_{\max}(B_0)+\|\widehat{\Delta B}\|_F]^2\sigma_{\max}(\text{Sh}\,K(\jmath\omega)) \qquad \text{(C7)}\end{aligned}$$

Finally, using Eqs. (C6) and (C7) it follows that

$$\begin{aligned}\min_{H\in\boldsymbol{H}}\lambda_{\min}\left[\jmath\,\text{Sh}\,\Phi^{-1}(H_1,\jmath\omega)+\jmath\,\text{Sh}\,F(H_2,\jmath\omega)\right] &\geq \min_k(1/\omega)(\Omega_{0,k}-\widehat{\Delta\Omega}_k)^2-\omega\\ &-\sigma_{\max}[\text{Sh}\,K(\jmath\omega)][\sigma_{\max}(B_0)+\|\widehat{\Delta B}\|_F]^2 \qquad \text{(C8)}\end{aligned}$$

Using a procedure similar to that given, it can be shown that

$$\begin{aligned}\min_{H\in\boldsymbol{H}}\lambda_{\min}[-\jmath\,\text{Sh}\,\Phi(H_1,\jmath\omega)-\jmath\,\text{Sh}\,F(H_2,\jmath\omega)] &\geq \min_k\omega-(1/\omega)(\Omega_{0,k}+\widehat{\Delta\Omega}_k)^2\\ &-\sigma_{\max}[\text{Sh}\,K(\jmath\omega)][\sigma_{\max}(B_0)+\|\widehat{\Delta B}\|_F]^2 \qquad \text{(C9)}\end{aligned}$$

Now, using Eqs. (C5), (C8), and (C9) and Theorem 3.1, the theorem is proved. □

Acknowledgments

This research was sponsored in part by the U.S. Air Force Office of Scientific Research under Contract F49620-92-C-0019 and the National Science Foundation under Grants ECS-91095588 and ECS-9496249.

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